PROBLEMS OF THE STRONG BENDING OF A THIN ELASTIC LAYER
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INTRODUCTION

It is noted in [1] that the elementary theory of bending, based on the Kirchhoff–Love hypothesis, is not applicable for the solution of a number of contact problems. A variant of the theory of bending can be constructed which is more general in comparison with the elementary theory and which allows of a correct formulation of the plane contact problem of the theory of elasticity. The problem discussed in the present article allows of "arbitrary" rotations, but, in the case where the curvature of one of the surfaces is given, remains linear.

§1. Let

$$\mathbf{r} = x\mathbf{e}_1, \ \mathbf{r}_* = \mathbf{r} + \mathbf{W},$$

where $\mathbf{r}$ and $\mathbf{r}_*$ are the radius-vectors of points of the lower surface of the layer before and after deformation, respectively; $\mathbf{W} = u_*(x)\mathbf{e}_1 + v_*(x)\mathbf{e}_2$ is the vector of the displacements; $\mathbf{e}_1$ and $\mathbf{e}_2$ are unit vectors along the coordinate axis. We define $\varepsilon_1(x)$ (the deformation of the lower surface) in the following manner:

$$\varepsilon_1 = \frac{1}{2}\left(\mathbf{r}_{*,1}^2 - \mathbf{r}_{,1}^2\right) = u_{*,1} + \frac{1}{2}u_{*,1}^2 + \frac{1}{2}v_{*,1}^2$$

(here and in what follows the subscript after the comma denotes differentiation with respect to the corresponding coordinate).

Let $\theta(x)$ be the angle between a tangent to the deformed lower surface and $\mathbf{e}_1$ (see Fig. 1); then

$$\operatorname{tg}\theta = v_{*,1}/(1 + u_{*,1}).$$

By $\mathbf{e}_{1*}$ and $\mathbf{e}_{2*}$ we denote the unit vectors of a tangent and a normal to the deformed lower surface. Then $\mathbf{e}_{1*} = \cos\theta\cdot\mathbf{e}_1 + \sin\theta\cdot\mathbf{e}_2$ and $\mathbf{e}_{2*} = -\sin\theta\cdot\mathbf{e}_1 + \cos\theta\cdot\mathbf{e}_2$. We note that $\mathbf{e}_{1*,1} = \theta,_1\mathbf{e}_2$, $\mathbf{e}_{2*,1} = -\theta,_1\mathbf{e}_{1*}$. With deformation of the layer, let the point M, determined with respect to the deformation by the radius-vector $\boldsymbol{\rho} = x\mathbf{e}_1 + y\mathbf{e}_2$, go over to the point M' (see Fig. 1), determined by the radius-vector

$$\boldsymbol{\rho}_* = \mathbf{r} + \mathbf{W} + (y + v)\mathbf{e}_{2*} + u\mathbf{e}_{1*}, \tag{1.1}$$

where $u = u(x, y)$; $v = v(x, y)$; $u(x, 0) \equiv 0$; and $v(x, 0) \equiv 0$.

Since the Kirchhoff–Love hypothesis is satisfied in the case $u(x, y) \equiv 0$ and $v(x, y)$ 0, we shall call u and v supplementary displacements inside the layer.

We assume the problem to admit of "arbitrary" rotations, but with elongations and the shears that are small in comparison with unity, and that the thickness of the layer is much less than the radius of curvature, i.e.,

$$\begin{gathered} h\theta_{,1} \sim \varepsilon, \text{ where } 1 + \varepsilon \approx 1, \\ \varepsilon_1 \sim \varepsilon, \quad \text{where } 1 + \varepsilon \approx 1, \\ \frac{\partial u}{\partial x}, \ \frac{\partial u}{\partial y}, \ \frac{\partial v}{\partial x}, \ \frac{\partial v}{\partial y} \sim \varepsilon, \text{ where } 1 + \varepsilon \approx 1. \end{gathered} \tag{1.2}$$

By ε_1^y, ε_2^y and ε_{12}^y we denote the deformations of the neighborhood of a particle. Then

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$$\varepsilon_1^y = \frac{1}{2}(\rho_{*,1}^2 - \rho_{,1}^2), \quad \varepsilon_2^y = \frac{1}{2}(\rho_{*,2}^2 - \rho_{,2}^2), \quad \varepsilon_{12}^y = \frac{1}{2}\rho_{*,1}\rho_{*,2}. \tag{1.3}$$

Using (1.1)-(1.3), we obtain

$$\varepsilon_1^y = \varepsilon_1 - y\theta_{,1} + \frac{\partial u}{\partial x} - v\theta_{,1}, \quad \varepsilon_2^y = \frac{\partial v}{\partial y}, \quad \varepsilon_{12}^y = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + u\theta_{,1}\right). \tag{1.4}$$

§2. Let (y_1, y_2) be a Cartesian system of coordinates and (x_1, x_2) an orthogonal curvilinear system of coordinates. The equations of equilibrium in the absence of mass forces assume the form

$$\frac{\partial(\sqrt{g}\,P^{\alpha\beta})}{\partial x^\alpha} + \sqrt{g}\,P^{\sigma\alpha}\Gamma_{\sigma\alpha}^\beta = 0 \qquad (\alpha, \beta, \sigma = 1, 2),$$

where $\Gamma_{\gamma\alpha}^\beta = {}^1/_2\, g^{\beta\omega}[(\partial g_{\omega\gamma}/\partial x^\alpha) + (\partial g_{\omega\alpha}/\partial x^\gamma) - (\partial g_{\gamma\alpha}/\partial x^\omega)]$ are Christoffel symbols of the second kind; $g_{\alpha\beta} = \mathbf{\partial}_\alpha \cdot \mathbf{\partial}_\beta$ are the components of a metric tensor; and $\mathbf{\partial}_\alpha = (\partial y^\beta/\partial x^\alpha)\mathbf{e}_\beta$ ($\beta = 1, 2$). Let us examine the curvilinear coordinates (x, y), connected with the curved lower surface of the layer (Fig. 2). As the coordinate x of the point M we take the Cartesian coordinate of the same point, which, with deformation, goes over into the base of a perpendicular dropped from M to the curve LN. The coordinate y is the distance from the point M to the curve LN. Then $g_{11} = (\sqrt{1 + 2\varepsilon_1} - y\theta_{,1})^2$, $g_{22} = 1$, and $G_{12} = 0$. Going over to the physical components of the tensor of the stresses $P^{\alpha\beta} = P_*^{\alpha\beta}\sqrt{g_{\alpha\alpha}g_{\beta\beta}}$ and denoting $\sigma_1 = P_*^{11}$, $\sigma_2 = P_*^{22}$, $\tau_{12} = P_*^{12}$, using (1.2) we obtain

$$\partial\sigma_1/\partial x + \partial\tau_{12}/\partial y - 2\theta_{,1}\tau_{12} = 0, \ \partial\tau_{12}/\partial x + \partial\sigma_2/\partial y - \theta_{,1}(\sigma_2 - \sigma_1) = 0. \tag{2.1}$$

Equations analogous to (1.4) and (2.1) were obtained in [2] for a description of the boundary layer.

§3. Let an elastic body occupy the volume of the curvilinear rectangle $\Omega = \{x, y \mid y \in [0, h],\ x \in [-l, l]\}$ (Fig. 3) and be in a state of plane deformation. Let the curvature of the lower surface be given, i.e., let the function $\theta_{,1}(x)$ be known. The problem consists in seeking functions $\sigma_1, \sigma_2, \tau_{12}, \varepsilon_1^y, \varepsilon_2^y, \varepsilon_{12}^y, \varepsilon_1$, u, and v satisfying Eqs. (1.4) and (2.1) and Hooke's law

$$\sigma_1 = \alpha\varepsilon_1^y + \beta\varepsilon_2^y, \quad \sigma_2 = \beta\varepsilon_1^y + \alpha\varepsilon_2^y, \quad \tau_{12} = 2\mu\varepsilon_{12}^y \tag{3.1}$$

for $\alpha = \lambda + 2\mu$, $\beta = \lambda$, $\lambda > 0$, and $\mu > 0$ inside the region Ω and the following relationships at the bounary of the region:

$$u|_{y=0} = v|_{y=0} = 0, \ \tau_{12}|_{y=0} = \tau_-, \tag{3.2}$$

$$v|_{y=h} = v_+ \ \text{ or } \ \sigma_2|_{y=h} = \sigma_{2+}, \ \tau_{12}|_{y=h} = \tau_+ \ \text{ or } \ u|_{y=h} = u_+;$$
$$\sigma_1|_{x=\pm l} = \sigma_{*\pm}, \ \tau_{12}|_{x=\pm l} = \tau_{*\pm}. \tag{3.3}$$

A specific characteristic of the problem is that although the use of Eqs. (1.4) and (2.1) admits of arbitrary rotations, since the function $\theta_{,1}(x)$ is given, the problem becomes linear. In view of the fact that the system of coordinates connected with the lower surface is deformed in an unknown manner (the equations contain the unknown function ε_1, i.e., the deformation of the lower surface — in the general case, a nonlinear function of the displacements of this surface), in distinction from the usual problem of the theory of elasticity, at the surface y = 0 an additional condition must be imposed: $u|_{y=0} = 0$.

§4. We assume that a solution of the problem exists. Multiplying (2.1) by u and v, respectively, and collecting and integrating with respect to Ω, we obtain

$$\int_0^h (\sigma_1 u + \tau_{12} v)\big|_{-l}^{l}\, dy + \int_{-l}^{l} (\tau_{12} u + \sigma_2 v)\big|_0^h\, dx = \tag{4.1}$$
$$= \int_\Omega \left[\alpha\left(\frac{\partial u}{\partial x}\right)^2 + \alpha\left(\frac{\partial v}{\partial y}\right)^2 + 2\beta\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + (\varepsilon_1 - y\theta_{,1})\left(\alpha\frac{\partial u}{\partial x} + \beta\frac{\partial v}{\partial y}\right)\right] d\Omega.$$

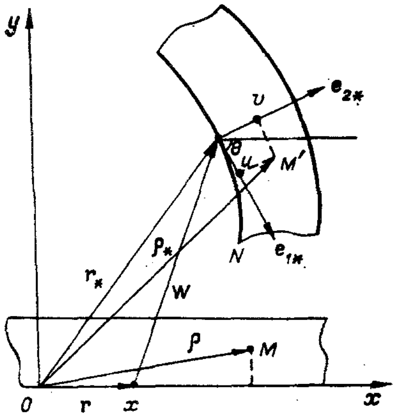

Fig. 1

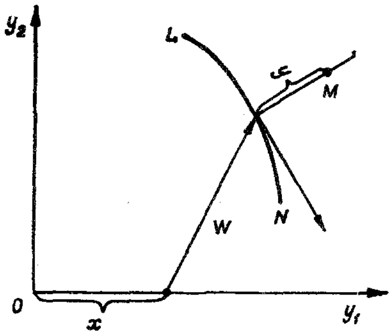

Fig. 2

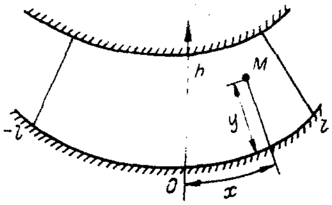

Fig.3

Integrating the first of Eqs. (2.1) with respect to y from 0 to h and using the boundary condition (3.3), we obtain

$$\varepsilon_1 = \frac{1}{2} h\theta_{,1} - \frac{1}{\alpha h}\int_0^h \left(\alpha \frac{\partial u}{\partial x} + \beta \frac{\partial v}{\partial y}\right) dy. \tag{4.2}$$

From (4.1) and (4.2) we obtain

$$\begin{aligned}
&\int_0^h (\sigma_1 u + \tau_{12} v)\big|_{-l}^{l}\, dy + \int_{-l}^{l} (\tau_{12} u + \sigma_2 v)\big|_0^h\, dx = \\
&= \int_\Omega \left[\alpha\left(\frac{\partial u}{\partial x}\right)^2 + \alpha\left(\frac{\partial v}{\partial y}\right)^2 + 2\beta \frac{\partial u}{\partial x}\frac{\partial v}{\partial y}\right] d\Omega + \int_\Omega \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 d\Omega + \int_\Omega \left(\frac{h}{2} - y\right)\theta_{,1}\times \\
&\times\left(\alpha\frac{\partial u}{\partial x} + \beta\frac{\partial v}{\partial y}\right) d\Omega - \int_{-l}^{l}\left[\frac{1}{\alpha h}\left(\int_0^h\left(\alpha\frac{\partial u}{\partial x} + \beta\frac{\partial v}{\partial y}\right)dy\right)^2\right] dx.
\end{aligned} \tag{4.3}$$

We note that

$$\int_{-l}^{l}\left[\frac{1}{\alpha h}\left(\int_0^h\left(\alpha\frac{\partial u}{\partial x} + \beta\frac{\partial v}{\partial y}\right)dy\right)^2\right] dx \leqslant \int_\Omega\left[\alpha\left(\frac{\partial u}{\partial x}\right)^2 + 2\beta\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \frac{\beta^2}{\alpha}\left(\frac{\partial v}{\partial y}\right)^2\right] d\Omega. \tag{4.4}$$

Thus, we have satisfied the energy identity, from which the singularity of the solution of the problem can be shown. In actuality, let there exist two solutions of the problem. We denote the difference of these solutions by the corresponding letter. Taking into consideration that the solutions coincide at the boundary of the region, from (4.3) we obtain

$$\int_\Omega \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 d\Omega + \int_{-l}^{l}\left\{\left[\alpha\left(\frac{\partial u}{\partial x}\right)^2 + \alpha\left(\frac{\partial v}{\partial y}\right)^2 + 2\beta\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}\right] dy - \frac{1}{\alpha h}\left(\int_0^h\left(\alpha\frac{\partial u}{\partial x} + \beta\frac{\partial v}{\partial y}\right)dy\right)^2\right\} dx = 0. \tag{4.5}$$

In view of (4.4), the second term in (4.5) is nonnegative; consequently,

$$0=\int_{-l}^{l}\left\{\int_{0}^{h}\left[\alpha\left(\frac{\partial u}{\partial x}\right)^{2}+\alpha\left(\frac{\partial v}{\partial y}\right)^{2}+2\beta\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}\right]dy-\right.$$
$$\left.-\frac{1}{\alpha h}\left(\int_{0}^{h}\left(\alpha\frac{\partial u}{\partial x}+\beta\frac{\partial v}{\partial y}\right)dy\right)^{2}\right\}dx\geqslant\int_{\Omega}\frac{\alpha^{2}-\beta^{2}}{\alpha}\left(\frac{\partial v}{\partial y}\right)^{2}d\Omega; \tag{4.6}$$

$$\partial u/\partial y+\partial v/\partial x\equiv 0. \tag{4.7}$$

Since $(\alpha^2 - \beta^2)/\alpha > 0$,

$$\partial v/\partial y\equiv 0. \tag{4.8}$$

Substituting (4.8) into (4.6), we obtain ($y = hz$)

$$\int_{-l}^{l}\left[\int_{0}^{h}\left(\frac{\partial u}{\partial x}\right)^{2}dy-\frac{1}{h}\left(\int_{0}^{h}\frac{\partial u}{\partial x}dy\right)^{2}\right]dx=0. \tag{4.9}$$

We represent $\partial u/\partial x$ in the form of a Fourier series in terms of a Legendre polynomial, orthogonal to [0, 1]:

$$\frac{\partial u}{\partial x}=\sum_{i=0}^{\infty}c_i(x)P_i(z).$$

From (4.9) we obtain $\sum_{i=1}^{\infty}\frac{1}{2i+1}c_i^2=0$; consequently, $c_i \equiv 0, \forall i = 1, \ldots$ Thus, $\partial u/\partial x = c_0 =$ const, but, from (3.2), $\partial u/\partial x|_{y=0}$; consequently, $\partial u/\partial x \equiv 0$. Together with (4.7) and (4.8), this shows the singularity of the solution of the problem.

§5. It is proposed to solve the problem by a method described in [3], using an expansion of unknown functions in series in terms of Legendre polynomials. The first approximation is as follows:

$$u=u_0P_0+u_1P_1+u_2P_2+u_3P_3,$$
$$v=v_0P_0+v_1P_1+v_2P_2,$$

$$\sigma_1=d_0P_0+d_1P_1,\ \sigma_2=q_0\ P_0+q_1P_1,\ \tau_{12}=s_0P_0+s_1P_1+s_2P_2,$$

where $P_i(z)$ are Legendre polynomials, orthogonal in [0, 1]:

$$P_i(z)=\frac{1}{i!}\frac{d^i}{dz^i}[z^i(z-1)^i],\qquad z=y/h.$$

The following equations are satisfied:

$$\int_0^1\left(\frac{\partial\tau_{12}}{\partial x}+\frac{1}{h}\frac{\partial\sigma_2}{\partial z}\right)P_0dz=0,\quad \int_0^1\left(\frac{\partial\sigma_1}{\partial x}+\frac{1}{h}\frac{\partial\tau_{12}}{\partial z}\right)P_kdz=0\qquad (k=0;1),$$
$$\int_0^1(\sigma_1-\alpha\varepsilon_1^y-\beta\varepsilon_2^y)P_kdz=0,\quad \int_0^1(\sigma_2-\beta\varepsilon_1^y-\alpha\varepsilon_2^y)P_kdz=0,$$
$$\int_0^1(\tau_{12}-2\mu\varepsilon_{12}^y)P_kdz=0\qquad (k=0,1,2).$$

The deformations ε_1^y, ε_2^y, and ε_{12}^y are taken in the form

$$\varepsilon_1^y=\left(\varepsilon_1-\frac{1}{2}h\theta_{,1}\right)P_0-\frac{1}{2}h\theta_{,1}P_1+\frac{\partial}{\partial x}(u_0P_0+u_1P_1),$$
$$\varepsilon_2^y=\frac{1}{h}\frac{\partial}{\partial z}(v_0P_0+v_1P_1+v_2P_2),$$
$$2\varepsilon_{12}^y=\frac{\partial}{\partial x}(v_0p_0)+\frac{1}{h}\frac{\partial}{\partial z}(u_0P_0+u_1P_1+u_2P_2+u_3P_3).$$

The boundary conditions at the surfaces y = 0 and y = h give the following equations:

$$u_0 - u_1 + u_2 - u_3 = 0,\ v_0 - v_1 + v_2 = 0,\ s_0 - s_1 + s_2 = \tau_{-},$$
$$v_0 + v_1 + v_2 = v_+ \quad \text{or} \quad q_0 + q_1 = \sigma_{2+},$$
$$s_0 + s_1 + s_2 = \overset{*}{\tau}_{12+} \quad \text{or} \quad u_0 + u_1 + u_2 + u_3 = u_+.$$

Thus, the problem (1.4), (2.1), (3.1), (3.3) is reduced to a boundary-value problem for ordinary differential equations with the following boundary conditions:

$$d_0|_{x=\pm l} = d_{0\pm},\ d_1|_{x=\pm l} = d_{1\pm},\ s_0|_{x=\pm l} = s_{0\pm}.$$

§6. By way of example, let us consider the problem of the compression of an elastic layer by rigid round cylindrical dies. Let a layer of length $2l$ and thickness h be compressed by dies of constant curvature $\theta_{,1}(x) = \varkappa =$ const. We assume that contact is established between the upper and lower surfaces of the layer and the dies. In view of the symmetry of the problem with respect to x = 0, we seek its solution in the region

$$\Omega = \{x,\ y | x \in [0,\ l],\ y \in [0,\ h]\}.$$

We require that the unknown functions take on the following values at the boundary of Ω:

$$u|_{y=0} = v|_{y=0} = \tau_{12}|_{y=0} = 0,\ \tau_{12}|_{y=h} = 0,\ v|_{y=h} = k_1 h,$$

where k_1 = const, $k_1 < 0$,

$$\sigma_1|_{x=l} = \tau_{12}|_{x=l} = u|_{x=0} = \partial v/\partial x|_{x=0} = 0.$$

Without breaking down the generality, we assume that $\nu = 0.25$ or, what is the same thing, $\lambda = \mu$.

The use of the above-described procedure of expansion in terms of Legendre polynomials reduces the problem to a boundary-value problem for a system of linear differential equations with constant coefficients,

$$\begin{gathered} d_{0,11} = 0, \\ 6u_{1,11} - 11\frac{2}{h}v_{0,1} - 5\frac{4}{h^2}u_1 = 0, \\ \frac{5}{6}v_{0,11} + \frac{11\cdot 2}{6h}u_{1,1} - 9\frac{4}{h^2}v_0 = \varkappa - \frac{18k_1}{h} \end{gathered} \qquad (6.1)$$

(the remaining unknown functions are uniquely expressed in terms of d_0, u_1, and v_0), with five boundary conditions

$$\begin{gathered} u_1|_{x=0} = v_0|_{x=0} = 0,\ d_0|_{x=l} = 0, \\ \left(\frac{2}{h}u_1 + v_{0,1}\right)\Big|_{x=l} = 0, \quad \left(u_{1,1} - \frac{2}{h}v_0\right)\Big|_{x=l} = \frac{\varkappa h}{2} - k_1. \end{gathered} \qquad (6.2)$$

Substituting the solution of (6.1), found with an exactness up to the undetermined coefficients in (6.2), we have

$$\sigma_2|_{y=h} = \mu\frac{\varkappa h}{2}\left[-\frac{8}{3}k_* + \frac{8}{3(q_3 - q_4)}\left(q_3 e^{\frac{2q_3(x-l)}{h}} - q_4 e^{\frac{2q_4(x-l)}{h}}\right)\right]; \qquad (6.3)$$

$$\sigma_2|_{y=0} = \mu\frac{\varkappa h}{2}\left[-\frac{8}{3}k_* - \frac{8}{3(q_3 - q_4)}\left(q_3 e^{\frac{2q_3(x-l)}{h}} - q_4 e^{\frac{2q_4(x-l)}{h}}\right)\right], \qquad (6.4)$$

where $k_* = -(2/\varkappa h)k$.

The stresses $\sigma|_{2\ y=0}$ and $\sigma_2|_{y=h}$ are represented graphically in Fig. 4. The point x_{**} in Fig. 4 corresponds to the maximal compressive stress at the surface y = h and to a minimal stress at $y = 0(l - x_{**})/l = h/l \cdot \ln(q_3/q_4)/(q_3 - q_4)$, i.e., x_{**} approaches l with an increase in the ratio of the length l to the thickness h.

From the statement of the problem it is clear that the stresses $\sigma_2|_{y=0}$ and $\sigma_2|_{y=h}$ must be compressive over the whole surface, i.e.,

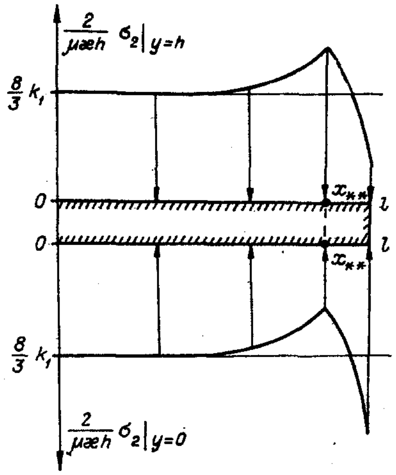

Fig. 4

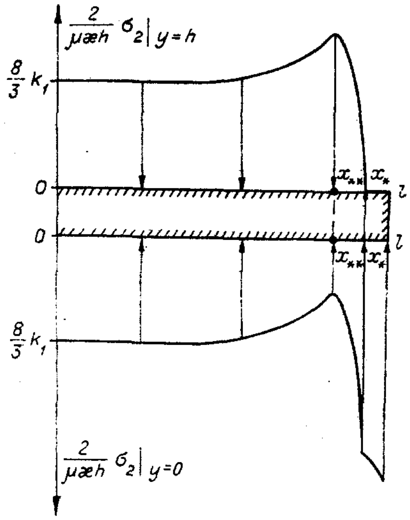

Fig. 5

$$\sigma_2|_{y=0} \leqslant 0, \; \sigma_2|_{y=h} \leqslant 0. \tag{6.5}$$

If one of the conditions (6.5) breaks down at some part of the surface (there is a "departure" of the corresponding part of the surface from the rigid die), the boundary conditions must be formulated in another form; that is, it must be assumed that the corresponding part of the surface is free of stresses. In this case, the condition that the elastic layer "does not enter the die" must be satisfied, i.e.,

$$v|_{y=0} \geqslant 0, \; v|_{y=h} \leqslant k_1 h. \tag{6.6}$$

In [1], it is shown that the conditions (6.5) and (6.6) make it possible to obtain a unique solution of the contact problem with an unknown contact zone. From (6.3) and (6.4) it can be seen that the condition (6.5) is satisfied for $-k_1 \geqslant \varkappa h/2$. We note that for $-k_1 < \varkappa h/2$ the condition (6.5) breaks down at the surface y = h and close to the end of the layer, which means that for $-k_1 < \varkappa h/2$ the ends of this surface depart from the die. We therefore formulate the problem in the following manner.

In the region $\Omega_I = \{x, y | x \in [0, x_*], y \in [0, h]\}$ let the unknown functions satisfy the Eqs. (1.4), (2.1), and (3.1) and at the surfaces y = 0, y = h, and x = 0 let them take on the values

$$u^I|_{y=0} = v^I|_{y=0} = \tau_{12}^I|_{y=0} = 0, \quad \tau_{12}^I|_{y=h} = 0, \; v^I|_{y=h} = k_1 h,$$
$$u^I|_{x=0} = \partial v^I/\partial x|_{x=0} = 0,$$

and in the region $\Omega_{II} = \{x, y | x \in [x_*, l], y \in [0, h]\}$ let them satisfy the same equations and the following relationships at the surfaces y = 0, y = h, and x = l:

$$u^{II}|_{y=0} = v^{II}|_{y=0} = \tau_{12}^{II}|_{y=0}| = \tau_{12}^{II}|_{y=h} = \sigma_2^{II}|_{y=h} = 0,$$
$$\sigma_1^{II}|_{x=l} = \tau_{12}^{II}|_{x=l} = 0.$$

In addition, the conditions of conjugation are satisfied at $x = x_*$:

$$u^I = u^{II}, \; v^I = v^{II}, \; \sigma_1^I = \sigma_1^{II}, \; \tau_{12}^I = \tau_{12}^{II}.$$

Applying the procedure of the expansion of the function in terms of Legendre polynomials we reduce the problem to a boundary-value problem for two systems of linear equations with constant coefficients of the fifth order each,

$$d_{0,1}^I = 0,$$
$$6u_{1,11}^I - 11\frac{2}{h}v_{0,1}^I - 5\frac{4}{h^2}u_1^I = 0, \tag{6.7}$$
$$\frac{5}{6}v_{0,11}^I + \frac{11}{6}\frac{2}{h}u_{1,1}^I - 9\frac{4}{h^2}v_0^I = \varkappa - \frac{18k_1}{h};$$

$$
\begin{gathered}
d_{0,1}^{\mathrm{II}} = 0, \\
192\, u_{1,11}^{\mathrm{II}} - 223\, v_{0,1}^{\mathrm{II}} \frac{2}{h} - 175 \frac{4}{h^2} u_1^{\mathrm{II}} = 0, \\
175\, v_{0,11}^{\mathrm{II}} + 223\, u_{1,1}^{\mathrm{II}} \frac{2}{h} - 432 \frac{4}{h^2} v_0^{\mathrm{II}} = \frac{\varkappa h}{2},
\end{gathered} \tag{6.8}
$$

with the five boundary conditions

$$
\begin{gathered}
u_1^{\mathrm{I}}|_{x=0} = v_{0,1}^{\mathrm{I}}|_{x=0} = 0, \quad d_0^{\mathrm{II}}|_{x=l} = 0, \\
\left(\frac{2}{h} u_1^{\mathrm{II}} + v_{0,1}^{\mathrm{II}}\right)\Big|_{x=l} = 0, \quad \left(4u_{1,1}^{\mathrm{II}} - \frac{2}{h} v_0^{\mathrm{II}}\right)\Big|_{x=l} = 4\frac{\varkappa h}{2},
\end{gathered}
$$

and five conditions of conjugation at $x = x_*$,

$$
\begin{gathered}
d_0^{\mathrm{I}} = d_0^{\mathrm{II}}, \ u_1^{\mathrm{I}} = u_1^{\mathrm{II}}, \ v_0^{\mathrm{I}} = v_0^{\mathrm{II}}, \\
d_1^{\mathrm{I}} = d_1^{\mathrm{II}}, \ s_0^{\mathrm{I}} = s_0^{\mathrm{II}}.
\end{gathered}
$$

The form of the system (6.7) is known, and the roots of the characteristic polynomial for the system (6.8) are equal to $\lambda_{1,2} = \pm\,(q_{*5} + iq_{*6})$, $\lambda_{3,4} = \pm\,(q_{*5} - iq_{*6})$, $q_{*5} = \sqrt{49/40}$, $q_{*6} = \sqrt{11/40}$, and $q_{*3,4} = 2(19 \pm 2\sqrt{34})/5$. The conditions (6.5) and (6.6) uniquely determine the unknown contact boundary x_*. In the selection of l/h there is a certain arbitrariness; for example, it can be assumed that $x_*/h = \pi + 2\pi n$. In this case,

$$
\frac{l - x_*}{x_*} = \frac{q_{*6}}{\pi (q_{*3} + q_{*4})} \frac{(1 - k_*)}{k_*}.
$$

The stresses $\sigma_2|_{y=0}$ and $\sigma_2|_{y=h}$ are shown in Fig. 5. As a result of the solution, the result is obtained that the stress $\sigma_2^{\mathrm{II}}|_{y=0}$ is proportional to $v^{\mathrm{II}}|_{y=h}$ and will be negative if the inequality (6.6) is satisfied. In the region Ω_{I},

$$
\begin{gathered}
\sigma_2|_{\substack{y=0\\y=h}} = \mu \frac{\varkappa h}{2}\left[-\frac{8}{3}k_* \mp \frac{1}{11}\left(\frac{45 - 43q_{*3}^2}{q_{*3}} c_1 e^{\frac{2q_{*3}(x - x_*)}{h}} + \frac{45 - 43q_{*4}^2}{q_{*4}} c_2 e^{\frac{2q_{*4}(x - x_*)}{h}}\right)\right], \\
c_1 = -\frac{(q_{*4}^2 + 1)}{3q_{*4}(q_{*3}^2 - q_{*4}^2)}\left[4k_*(3 + q_{*4}^2) - (1 - k_*)(9k_* - q_{*4}^2)\right], \\
c_2 = \frac{(q_{*3}^2 + 1)}{3q_{*3}(q_{*3}^2 - q_{*4}^2)}\left[4k_*(3 + q_{*3}^2) - (1 - k_*)(9k_* - q_{*3}^2)\right].
\end{gathered}
$$

In Fig. 5, x_{**} is the point of the least and greatest compressive stress at the surfaces $y = h$ and $y = 0$, respectively. From the equality

$$
(x_* - x_{**})/x_* = (h/x_*)A
$$

(A is a constant expressed in terms of q_3 and q_4), determining the point x_{**}, it follows that x_{**} approaches x_* with a rise in the ratio x_*/h and, consequently, of l/h. The inequality (6.5) is satisfied for $k_* \geqslant 0.14$. For $k_* \approx 0.14$, the "zone of departure" $(l - x_*)/l$ will be maximal and equal to $\approx 1/(1+2n) \cdot 20\%$.

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